Paradox Machines

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Source of Mathematics

Where do the laws of mathematics come from?

The set of “known” mathematical laws has evolved over time (has a history), due to:

- Discovery (math has transcendent status as *Truth*)
- Development of new practical techniques (math is a set of *tools*)

Ontological status of mathematical law is a philosophical issue...

Status of Mathematical Law

As mathematical techniques have evolved, so has our philosophy of mathematics:

- Philosophical understanding of mathematical truth underlies accepted mathematical law (e.g. axioms)
- Technical results may require a shift in philosophy

Many mathematicians have labored to provide a satisfactory philosophy of mathematics (are/were philosophers).

Paradoxes

Paradoxes are centerpieces of the historical tension between philosophy and technique.

Paradox: an apparent contradication of facts (inconsistency) in *:

- Logic/technique of reasoning system itself
- Semantics of reasoning system

In any reasoning system, the discovery of a paradox:

- May require technical resolution†
- Challenges philosophy

*Distinction originally due to Ramsey (1926)

†*Any* logical inconsistency in a system of reasoning renders it meaningless.

Epimenide's Liar Paradox

The oldest known paradox is the Liar Paradox of Epimenide's (~ 600 B.C.):

"This sentence is a lie"

- If the assertion is false, it is true
- If the assertion is true, it must be false

Note that self-reference is a primary culprit...

The Effect

Liar paradox apparently one of logic at a high level (phrased in natural language).

Simple, old, confounding:

> *’Twas the Liar made me die, and the bad nights caused thereby*.
> – Inscription on tombstone of Philetus of Cos (340-285 BC)

First formal resolution due to Gödel in 1934: equivalent assertion *“this sentence is in the set of true sentences”* logically meaningless.

Paradox of Rationality

To the ancient Greeks, mathematics = Truth, “all is number”. There exists some indivisible *Unit* (“the one”) which underlies all numbers.

This implies that any two magnitudes can be expressed as multiples of Unit, and so put in a whole-number ratio with one another...

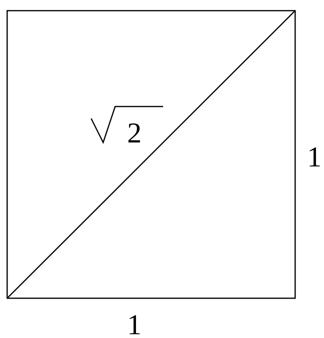
- *Rational numbers*: numbers that can be expressed as a whole-number ratio (whole numbers and fractions)
- *Rationality*: intelligible reality presupposes the primordial status of Unit*

* *“Order always means ratio.”* – Aristotle, Physics (VII)

Irrational Numbers Exist

Pythagoreans (or contemporaries, ~ 500 B.C.) discover that the proportion of the side of a square to its diagonal cannot be rational.

Noting the following*:



This is equivalent to observing that $\sqrt{2}$ is irrational. To the Greeks, this was an earth-shattering paradox.

*By Pythagoras' Theorem we have $1^2 + 1^2 = c^2$, i.e. $2 = c^2$ and $c = \sqrt{2}$.

The Effect

The Pythagoreans drown themselves, hide their discovery, etc.: myths.

The notion of “incommensurables” (numbers not expressible via ratio) accepted in mathematics, not developed (“unknowable”). Math = Truth still.

Paradox of philosophy, not logic; arithmetic on rationals remains consistent.

Zeno's Dichotomy

Zeno's Dichotomy (~ 500 B.C.): a paradox of the infinite divisibility of space and time.

Suppose an object travels from point A to point B:

- In order to traverse AB, it must traverse a distance equalling $AB/2$, and before that one equalling $AB/4$, before that $AB/8$, ...
- If space is infinitely divisible, this goes on ad infinitum

Therefore, if space is infinitely divisible, motion is not possible.

The Effect

Addressed by calculus, atomic physics 2,000 years later.

In antiquity, Aristotle distinguishes *potential* vs. *actual* reality:

- Time and space infinitely divisible in *mathematical* sense only
- *Infinite* divisibility has only *potential* status, no *actual* (physical) reality

"Time is not composed of individual nows any more than any other magnitude is composed of indivisibles" – Aristotle, Physics (V)

Paradox of Galileo

Galileo's *Due Nuoue Scienze* ("Two New Sciences", 1638) pioneers mathematical physics and experimental science.

Enumerates a sequence of paradoxes that "poke holes" in predominant Aristotelian philosophy of mathematics, including "Galileo's Paradox"...

Intuition: let $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$ and let $S = \{1, 4, 9, 16, ...\}$; then surely $\mathbb{N}$ is "larger" than S. But every square has a root and vice-versa:

$$(1, 1), (2, 4), (3, 9), (4, 16)...$$

Therefore, there are as many perfect squares as numbers.

The Effect

Two New Sciences an extended argument *for* experimentation and mathematical physics, *against* Aristotelian worldview, which assumes:

- Physical truth apprehendable only via “intellection”
- Potential-vs.-actual distinction (separation of physics and mathematics)

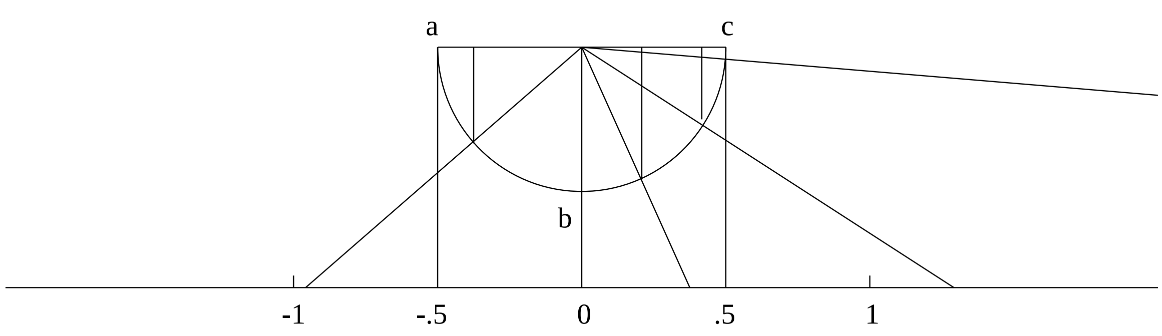
Galileo subverts authority by using finite techniques to reason about the infinite.

Instigates philosophical and technical investigations that lead to revolutionary developments...

Bolzano’s Paradox of the Infinite

Bolzano’s *Paradoxien des Unendlichen*, “Paradoxes of the Infinite” (1851), representative of this development.

Bolzano’s paradoxes *observations* on the strange behavior of infinities:



But Bolzano’s theory does not abhor a one-to-one correspondance between the real elements of $[0, 1]$ and $\mathbb{R}$.

The Effect

Refiguration of philosophy of the infinite promotes quantum leaps in technique:

- Theories of “infinitesmals” culminate in the Calculus*, real analysis
- Notion of one-to-one correspondance becomes standard technique for comparing “sizes” of sets
- Cantor’s transfinite arithmetic thoroughly accounts for “paradoxes”:
  - Infinite subsets of infinite sets may possess same *cardinality*
  - The cardinality of $\mathbb{R}$ greater than the cardinality of $\mathbb{N}$

*Compare e.g. acceleration as derivative of velocity to Aristotle

Foundations of Mathematics

In late 19th-early 20th century, *Foundations of mathematics* develops as major program of mathematics (and philosophy).

Program seeks a precise, primitive* account of numbers, arithmetic, and mathematical proof and meaning:

- Set theory
- Mathematical logic
- Computability theory

In these investigations, paradoxes abound...

*In the sense of first principles

Russell's Paradox

In 1901, Russel communicates his famous paradox of Cantorian set theory to Frege, who was axiomatizing the latter.

Russel's paradox is obtained via construction of the following set:

$$russelpdx = \{s \mid s \notin s\}$$

and asking, is $russelpdx \in russelpdx$?

- If $russelpdx \in russelpdx$, then $russelpdx \notin russelpdx$.
- If $russelpdx \notin russelpdx$, then $russelpdx \in russelpdx$.

Therefore, $russelpdx \in russelpdx$ if and only if $russelpdx \notin russelpdx$.

The Effect

Earth shattering effect on set theory; a logical paradox implying its inconsistency; required solution:

- Russell develops *ramified type theory* for sets
- Zermelo proposes Axiom Schema of Separation

Zermelo-Fraenkel set theory (ZF) preponderates; all approaches essentially eliminate “vicious circles” via *disallowing self reference*.

Russell’s Paradox cannot be constructed in ZF, but Separation considered unsatisfactorily restrictive*.

*See *anti-well foundedness axiom* proposed by Barwise.

Mathematical Logic

Mathematical logic introduced by Frege in *Begriffsschrift* (1879), attempts to capture the "laws of thought that transcend all particulars".

Logic is a well-defined deduction system for formulae ϕ constructed from "atomic" elements a and logical connectives:

$$\phi ::= a \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi$$

Formulae *provable* via deduction rules, e.g. from ϕ and $\phi \rightarrow \phi'$, deduce ϕ'.

First Order Logic

Logical *quantifiers* $\forall$, $\exists$ also components of formulae in *first-order* logic (FOL)*.

FOL so called because $\forall$, $\exists$ range over atomic elements (e.g. numbers in logic of arithmetic), not arbitrary formulae.

FOL is logic associated with Gödel's completeness and incompleteness theorems.

*Unquantified ϕ *propositional formulae*

Church's λ-Calculus: Second Order Logic

In 1930, Alonzo Church proposes the λ-calculus as a formalization of higher-order logic.

In *higher-order* logic, we can allow assertions to be abstract with respect to other assertions, e.g.:

> "An assertion is paradoxical iff* it implies its own contradiction"†

*iff = if and only if; we will also write $\iff$ for iff.

†We refer to this assertion as P_{pdx}.

λ-Calculus

For simplicity, we imagine the terms $M, N, P, \ldots$ of the λ-calculus as containing propositional formulae, plus λ-*abstraction* and *application*:

$$\lambda X.M \qquad\qquad MN$$

These terms allow second-order predicates to be defined, and to be asserted for particular cases:

- λ-abstraction is a higher-order universal quantifier ($\lambda \sim \forall$)
- Application means that predicate M holds for particular elements N ("M is a property of N")

λ-Calculus

The λ calculus was also equipped with a notion of logical equivalence that took into account the following intuition:

> An abstract predicate P applies to an object M iff all the conditions in P hold in the particular case of M *

Hence:

$M[N/X]$ means "substitute N for X in M"

$$(\lambda X.M)N \iff M[N/X]$$

*E.g. "the $russelpdx$ construction is paradoxical" follows by application of P_{pdx} to the construction.

Kleene-Rosser Paradox

Consider $(\lambda X.\neg XX)$, "given some X, it is not the case that X applies to itself". Let $kleenepdx = (\lambda X.\neg XX)(\lambda X.\neg XX)$.

The paradox is constructed by deduction in the logic*:

$$(\lambda X.\neg XX)(\lambda X.\neg XX) \iff (\neg XX)[(\lambda X.\neg XX)/X]$$

i.e.

$$(\lambda X.\neg XX)(\lambda X.\neg XX) \iff \neg(\lambda X.\neg XX)(\lambda X.\neg XX)$$

i.e.

$$kleenepdx \iff \neg kleenepdx$$

*Note the similarity to Russell's Paradox

The Effect

Paradox of logic, *pure* λ-calculus inconsistent as logic.

Type theory proposed to eliminate Kleene-Rosser paradox; typability of predicates eliminates possibility of self-reference*.

Typed λ-calculus, while consistent, significantly less expressive than λ-calculus.

Further, by shifting perspective, the existence of the paradox in the pure λ-calculus seems to have another, beneficial effect...

*Note similarity with resolution of Russell's Paradox.

Computability Theory

Computability theory is the study of functions that can be defined via some *algorithmic/effective* procedure.

In 1931, Alan Turing proposes Turing Machines (TMs) as a formal mathematical model of computation.

TMs become a benchmark of computability:

- A system is *Turing complete* iff it contains the functions that are computable by a TM

TMs = Pure λ calculus

In 1935, Kleene demonstrates that the *pure* λ-calculus, viewed as a formalization of computability, is Turing complete.

It is easily demonstrated that the *simply-typed* λ-calculus is *not* Turing complete:

- Types disallow self-reference
- Self-referencing *paradoxical combinators*[*] necessary to define recursive functions

Recursive definability essential to computability.

[*]Terminology due to Curry.

The Effect, Revisited

Paradox of *logic* turns out to be elemental to *computation*.

λ-calculus especially successful as computer programming language model generally, in addition to being the basis of Lisp, SML, OCaml, ...:

- Definition of functions via abstraction on formal arguments
- Evaluation of expressions via instantiation of formal with actual arguments

```
int f(int x) {...} // function abstraction
result = f(5);     // function application
```

Paradox Machines

Whence the paradox? From a *computational* point of view, paradoxical expressions manifest as *infinite loops*:

$$(\lambda X.\neg XX)(\lambda X.\neg XX) \Rightarrow \neg(\lambda X.\neg XX)(\lambda X.\neg XX) \Rightarrow$$
$$\neg\neg(\lambda X.\neg XX)(\lambda X.\neg XX) \Rightarrow \neg\neg\neg(\lambda X.\neg XX)(\lambda X.\neg XX) \Rightarrow \cdots$$

As a result of the Halting Problem, we note that infinite loops are a "necessary evil" of computation.

An aspect of λ-calculus that is *paradoxical* from a logical perspective is *essential* from a computational perspective*.

*Furthermore, mathematical semantics of pure λ-calculus due to Scott (1969).

Conclusion

Paradoxes are entertaining, but have deeper meaning:

- Incite advancements, challenge preassumptions
- Can they be eliminated?

In mathematical investigations, attention to underlying philosophies should be a component of reasoning.

> *"Every good mathematician is at least half a philsopher, and every good philosopher is at least half a mathematician".*
>
> – Gottlob Frege

Proof of the Irrationality of $\sqrt{2}$

Theorem 0.1 $\sqrt{2}$ *is not rational.*

Proof. Suppose on the contrary that $\sqrt{2}$ is rational. Then there exists p, q such that $\sqrt{2} = p/q$, which we'll assume are in lowest terms*. Thus $\sqrt{2} * q = p$ so $2 * q^2 = p^2$, implying that p^2 is even so that p must be even. But since p/q is in lowest terms, therefore q must be odd. Now, let $p = 2r$, so $2 * q^2 = (2r)^2$, therefore $2 * q^2 = 4 * r^2$, hence $q^2 = 2 * r^2$. But this implies q^2 is even, which means q is even, which is a contradiction. $\square$

*Without loss of generality.