6. Enumerating Anagrams / Computing Factorials

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1. Enumerating anagrams

2. Computing factorials with DrRacket

3. Astronomical numbers
Anagrams: Review

In CS 32, the word *anagram* signifies a permutation, or rearrangement of a group of letters or symbols. Thus the letters in the word *CAT* has *six* different anagrams:

- ACT
- ATC
- CAT
- CTA
- TAC
- TCA

(Note that only two of these are English words, the remaining four are jibberish, but are nevertheless anagrams, by our definition.)

The number of different anagrams of a word having $n$ different letters equals

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1.$$  

The notation $n!$ is called “$n$-factorial.”
Anagrams: Review

If the initial word has repeated letters, then the number of anagrams is reduced. Thus the word *MALL* has 12 anagrams:

```
ALLM   ALML   AMLL
LALM   LAML   LLAM   LLMA   LMAL   LMLA
MALL   MLAL   MLLA
```

With four letters, we might expect to find $4!$ permutations. However this number overcounts the correct value, because *LL* has only one anagram, not two. Thus we need to divide $4!$ by $2!$. Thus

$$\frac{4!}{2!} = \frac{24}{2} = 12.$$
Anagrams: Review

How many anagrams are in the word *LULL*?

How many are in *TOOT*?
Anagrams: Review

Recall, that a word consisting of $k$ different letters, with $r_1$ repetitions of the first, $r_2$ of the second, $\ldots$, and $r_k$ of the $k$-th letter, has

$$\frac{(r_1 + r_2 + \cdots + r_k)!}{r_1! \cdot r_2! \cdot \cdots \cdot r_k!}$$

unique anagrams. Thus the 11 letter word $ABRACADABRA$, which has 5 As, 2 Bs, 1 C, 1 D, and 2 Rs, has

$$\frac{11!}{5! \cdot 2! \cdot 1! \cdot 1! \cdot 2!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \cdot 2 \cdot 2} = \frac{11 \times 10 \times 9 \times 2 \times 7 \times 6}{5 \cdot 2} = 11 \times 10 \times 9 \times 2 \times 7 \times 6 = 99 \times 840 = 83,160$$

unique anagrams. *(Whenever one has a fraction with factorials appearing in both the numerator and denominator, one can always cancel common factors.)*
Computing Factorials

Computing large factorials though is difficult and time consuming, even with a calculator. Fortunately we can let DrRacket do the hard work for us. First though let’s examine the mathematical definition of the factorial: For any natural number\(^1 n\)

\[
n! = \begin{cases} 
1, & \text{if } n = 0, \\
 n \times (n - 1)!, & \text{otherwise.}
\end{cases}
\]

The statement that \(0! = 1\) is a useful convention. It might sound odd that there is exactly one way to permute an anagram that has zero letters. However, \(0! = 1^0\), as both are products that contain zero factors. And since, \(1^0 = 1\), it follows that \(0! = 1\).

The statement that \(n! = n \times (n - 1)!\), follows from observing that inside every factorial, e.g., \(5! = 5 \times 4 \times 3 \times 2 \times 1\), there is a slightly smaller one: Since \(4! = 4 \times 3 \times 2 \times 1\), we have \(5! = 5 \times 4!\).

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\(^1\) A natural number is a non-negative integer
Computing factorials in DrRacket

In the top (Definition) window, enter the following function definition.

```
(define (factorial n)
  (if (= n 0)
    1
    (* n (factorial (- n 1))))
```

Save the definition in a file called `factorial.rkt`. The special form “define” enables us to define new functions in Racket. Don’t worry that you don’t understand the syntax right away; it takes some getting used to. Can you observe any similarity between this function, and the mathematical definition on the previous slide?

Press the “Run” button (with the green running stick figure) on the upper right. Then you can evaluate expressions like

```
(factorial 7)  or  (factorial 52)
```

in the lower “Interaction” window. Just type in an expression after the command prompt “>” and press the return key on the keyboard.
DrRacket on a Macintosh

(define (factorial n)
  (if (= n 0)
      1
      (* n (factorial (- n 1)))))

Welcome to DrRacket, version 5.0.1 [3m].
Language: Pretty Big [custom].
> (factorial 7)
5040
> (factorial 52)
80658175170943878571660636856403766975289505440883277824000000000000
> |
Other ways to express the factorial

The previous definition of factorial is concise. The indentations and line breaks are for our benefit: DrRacket ignores them. We can also add comments (which begin with a semicolon) to help remind us how this function works. We can also introduce other helper functions if we like:

```scheme
;;; dec stands for decrement; it takes one away
;;; from its argument. So, (dec 3) => 2.
(define (dec n)
    (− n 1))

;;; ! returns the factorial of its argument, so (! 7) => 5040.
(define (! n)
    (if (zero? n)
        1 ; 0! = 1, otherwise
        (* n (! (dec n))))) ; n! = n * (n−1)!
```

The function zero? is already part of racket: (zero? 0) => #t, i.e.. “true,” and (zero? 1) => #f, that is, “false.”

Try copying the above into the Definition window in DrRacket, and run a few tests.
How many ways can one shuffle a deck of playing cards?

By the multiplication principle, there are $52$ choices for the first card, $51$ for the second, $50$ for the third, \ldots , up to $3$ choices for the last card. Thus, the number of shuffles is

$$D = 52 \times 51 \times 50 \times \cdots \times 3 = 80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000.$$ 

Outrageous Claim: This number is so large that the order of the cards in every well-shuffled deck has almost certainly never been realized before in the history of the universe!
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\approx 80 \times 10^{66}
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Some other huge numbers

- Number of seconds in a century $\approx 4.5 \times 10^9$.
- Number of human beings alive on Earth $\approx 6.7 \times 10^9$.
- Number of Oreo cookies sold between 1912 and 2007 $\approx 4.9 \times 10^{11}$.
- Federal debt (est. in US$ on 10/16/2010) $\approx 13.5 \times 10^{12}$.
- Number of seconds that have elapsed since the Big Bang $\approx 4 \times 10^{17}$.
- Number of distinct positions in a $3 \times 3 \times 3$ Rubik’s cube $\approx 4 \times 10^{19}$.
- Number of grains of sand on the earth $\approx 10^{21}$.
- Number of stars in the visible universe (Simon Driver, 2003) $\approx 7 \times 10^{22}$.
- Number of atoms in a human body $\approx 10^{28}$.
- Mass of the sun (in kilograms) $\approx 10^{30}$.
- Number of legal chess positions $\approx 10^{40}$.
- Number of distinct positions in a $4 \times 4 \times 4$ Rubik’s Revenge $\approx 10^{45}$.
- Number of permutations of 52 playing cards $\approx 8 \times 10^{67}$.
- Number of distinct positions in a $5 \times 5 \times 5$ Rubik’s Ultimate Cube $\approx 10^{74}$.
- Number of physical particles in the universe (inflationary model) $\approx 10^{80}$.
- Number of distinguishable games of go $\approx 10^{768}$.
Imagine this . . .

Imagine if every star in the universe has a life supporting planet that has been continuously populated with 7 billion inhabitants who have been shuffling 52-card decks of playing cards once every second since the universe was created. This would account for

\[(7 \times 10^{22} \text{ planets}) \times (7 \times 10^9 \text{ persons/planet}) \times (4 \times 10^{17} \text{ shuffles/person}) \]
\[= 2 \times 10^{50} \text{ shuffles.}\]

This equals only

\[\frac{2 \times 10^{50}}{8 \times 10^{67}} = 2.5 \times 10^{-18},\]

as a fraction of the total number of 52-card deck orderings. Thus the probability that a well shuffled deck has been realized before in the history of the universe is essentially 0.