One-dimensional Peg Solitaire

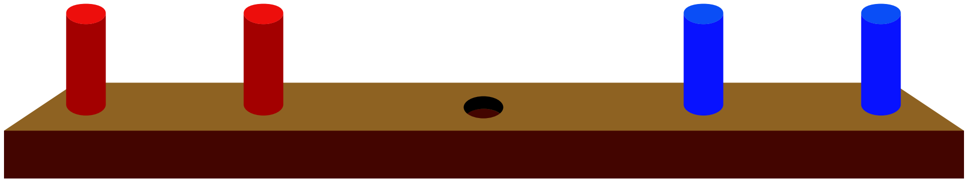
Lecture Notes for CS 32

Delivered on November 21, 2005

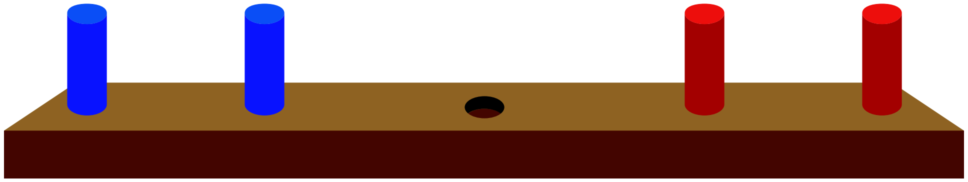
The Puzzle

- Five holes in a row.
- Two red pegs.
- Two blue pegs.

The Initial State

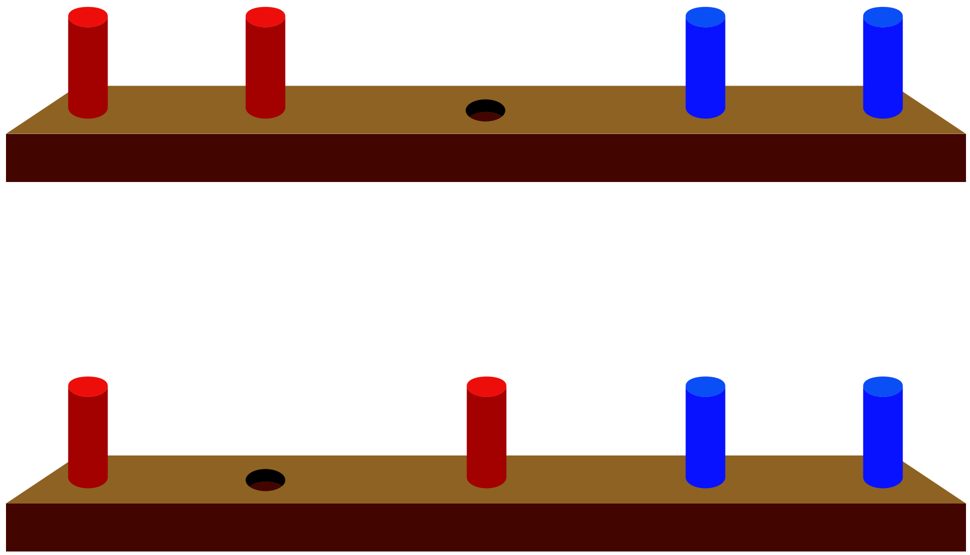


The Goal State



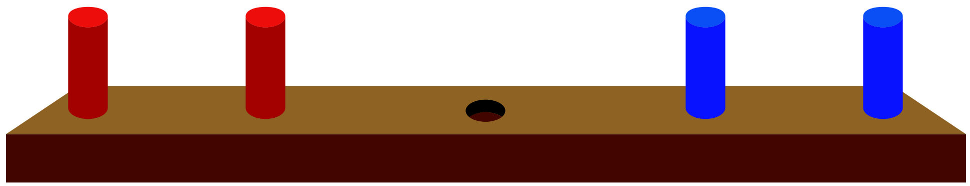
Four possible moves

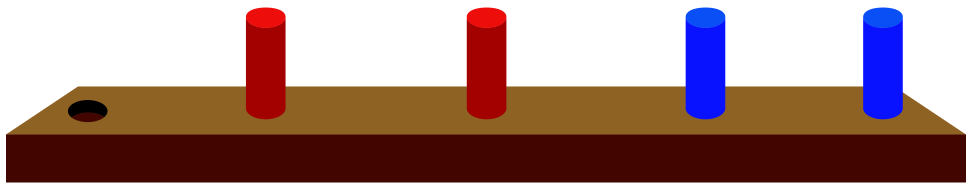
1. A red peg can *step* one position to the *right* into the hole.



Four possible moves

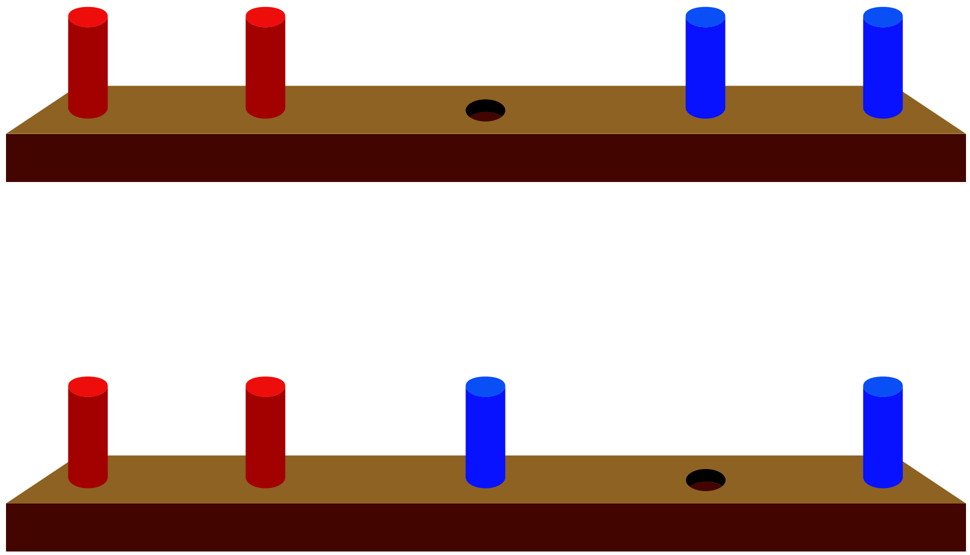
2. A red peg can *jump* two positions to the *right* into the hole.





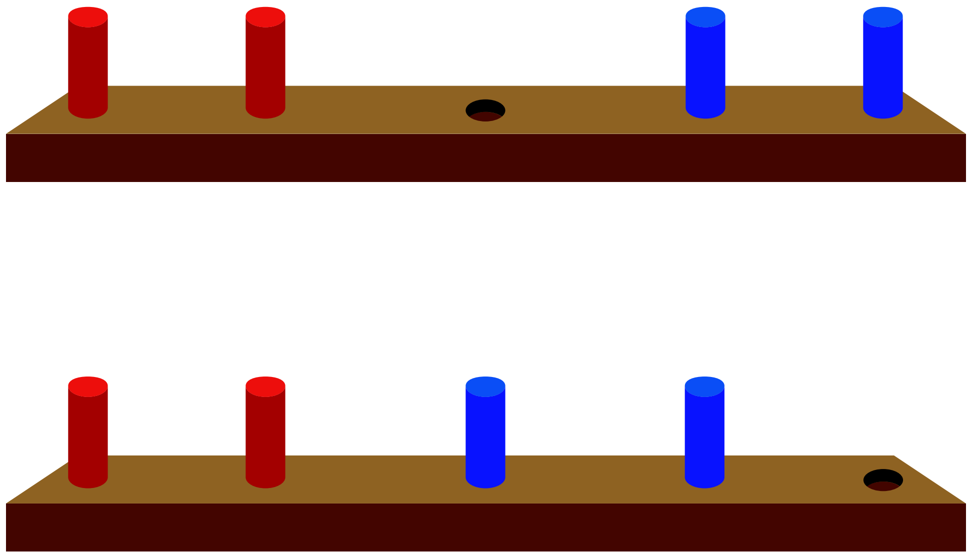
Four possible moves

3. A blue peg can *step* one position to the *left* into the hole.



Four possible moves

4. A blue peg can *jump* two positions to the *left* into the hole.



Questions

1. How many states (i.e. peg configurations) exist?
2. Which peg configurations can be obtained using a sequence of legal moves?
3. Is the goal configuration *accessible*? (Is it possible to solve this puzzle?)
4. If the puzzle can be solved, how many solutions exist?
5. If the puzzle can be solved, what is the shortest solution?
6. Can the puzzle be generalized?
7. Can you think of any other questions?

Data Representation

We can use a *list* to represent each state of the puzzle.

- A red R will represent a red peg.
- A blue B will represent a blue peg.
- An black H will represent the empty hole.
- A list of five symbols will represent the state.

The initial state is thus

(R R H B B).

The four successors are

(H R R B B), (R H R B B), (R R B H B), (R R B B H).

The goal state is thus

(B B H R R).

A Fussy but Important Observation

For every unique state of the four-peg puzzle, there exists a unique list containing two Rs, two Bs, and one H.

Thus there exists a *one-to-one correspondence* between the set of states of the puzzle, and the set of lists (containing two Rs, two Bs, and one H).

The Four Operations

Each of the four legal operations can be expressed symbolically:

Assuming we begin in the initial state:

1. Red Step

$$(R\ R\ H\ B\ B) \rightarrow (R\ H\ R\ B\ B).$$

2. Red Jump

$$(R\ R\ H\ B\ B) \rightarrow (H\ R\ R\ B\ B).$$

3. Blue Step

$$(R\ R\ H\ B\ B) \rightarrow (R\ R\ B\ H\ B).$$

4. Blue Jump

$$(R\ R\ H\ B\ B) \rightarrow (R\ R\ B\ B\ H).$$

The Four Operations: Generalization

Let the symbol ? denote a *single* stationary peg of either color, and let the symbol * denote *any number* of stationary pegs.

1. Red Step

$$(* \; R \; H \; *) \rightarrow (* \; H \; R \; *). \qquad \text{(RS)}$$

2. Red Jump

$$(* \; R \; ? \; H \; *) \rightarrow (* \; H \; ? \; R \; *). \qquad \text{(RJ)}$$

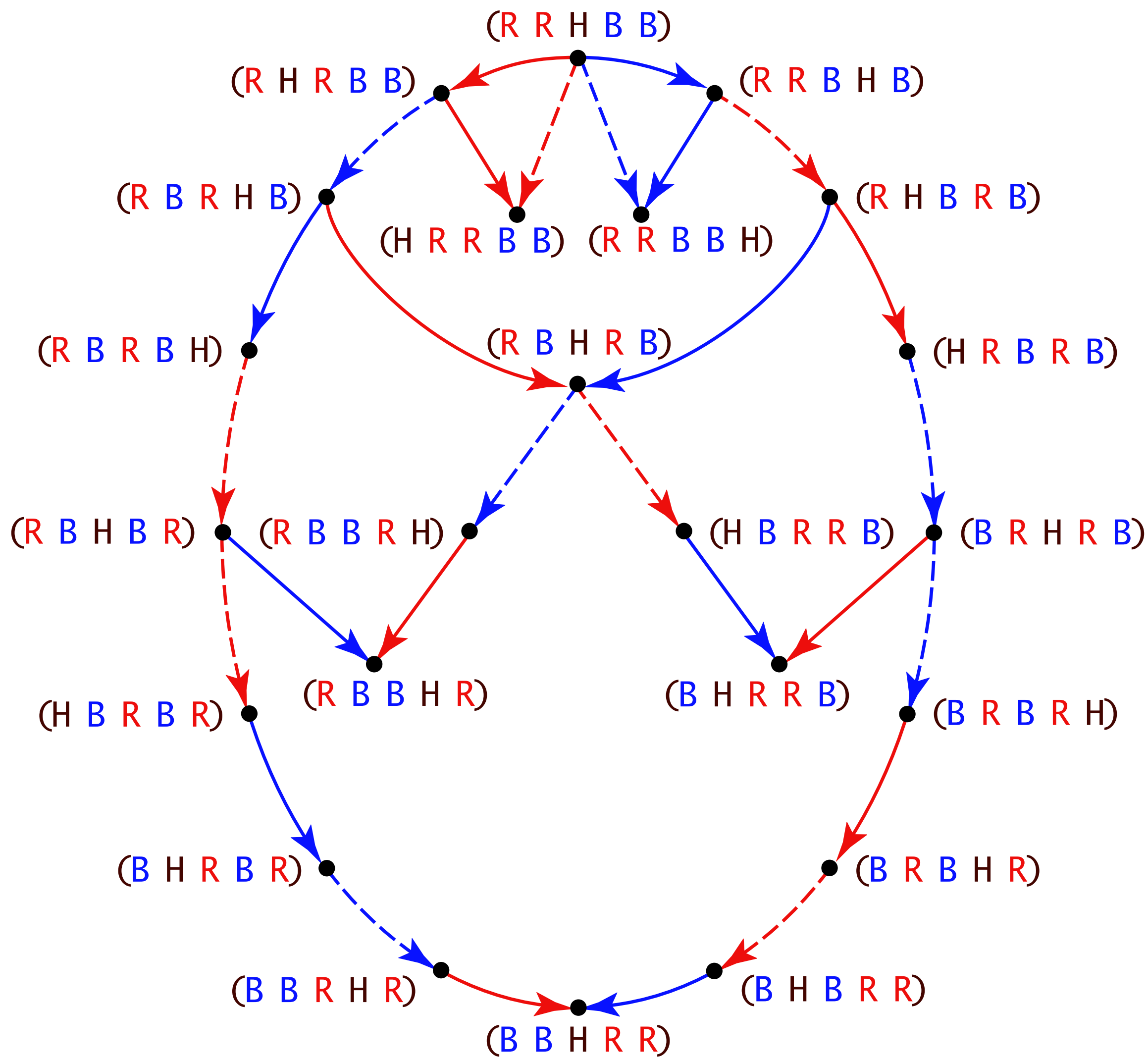
3. Blue Step

$$(* \; H \; B \; *) \rightarrow (* \; B \; H \; *). \qquad \text{(BS)}$$

4. Blue Jump

$$(* \; H \; ? \; B \; *) \rightarrow (* \; B \; ? \; H \; *). \qquad \text{(BJ)}$$

State Transistion Graph: A directed graph



Solid arrows denote steps; dashed arrows denote jumps.

How many states does the four-peg puzzle have?

- The graph indicates that there are 23 accessible states.
- There are some configurations though that that do not appear in the graph: e.g., (HBBRR).
- How many different peg configurations are there?

Solution Outline

1. Observe that the problem can be broken up into stages.
2. Multiplication Principle.

Multiplication Principle

(See, page 19 of Averbach and Chenin's, *Problem Solving Through Recreational Mathematics*, Dover, NY, 2000.)

Multistage Constructions

- Building an ice cream cone:
  1. Select the cone (cake, sugar, or waffle),
  2. Select the first scoop,
- Two Stage If a counting problem can be decomposed into a number of stages,problem can be decomposed into a

Number of ways of placing two red pegs into five holes:

$$\binom{5}{2} = \frac{5!}{2!\,3!} = 10.$$

Number of ways of placing two blue pegs into the remaining three holes:

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3.$$

Total number is the product

$$\binom{5}{2} \times \binom{3}{2} = 30.$$

How many solutions are there?

- Two solutions.
- How many steps are required to solve the *four*-peg puzzle?
- How many steps are required to solve an *eight*-peg puzzle?
  A $2n$ peg puzzle?

Computing the length of each solution

It is interesting to note that one can infer the length of each solution to the four-peg puzzle (and indeed, the more general $2n$-peg puzzle), without discovering the explicit sequence of operations.

We first determine the distance between the initial and the goal configurations. Since each peg must travel a distance of 3 holes, the total distance $D = 4 \times 3 = 12$.

Now note that a change in the order between a red peg and a blue peg requires a single jump. Likewise, to change the order of each red peg with *two* blue pegs requires *two* jumps. Therefore, the total number of jumps requires is $J = 2 \times 2 = 4$.

To complete the calculation, we note that each jump moves a peg by two holes, and each step by one hole. Thus $D = 2J + S$, whence $S = D - 2J = 12 - 2(4) = 4$.

Thus, the total number of moves required is $M = J + S = 4 + 4 = 8$.