Function countOne(n)
If n=1 Or n=0    //base case
    Return n
Else if n is even
    countOne(n/2)    //recursive
Else if n is odd
    countOne(n/2) +1    //recursive
End
2 a

\[ \sum_{i=1}^{N} (2i - 1) = 2 \sum_{i=1}^{N} i - \sum_{i=1}^{N} 1 = \frac{2N(N + 1)}{2} - N = N^2 \]

b. Proof by induction

Base case (N = 1): The formula holds. (Both sides equal)

Inductive case: Assume the formula holds for N=k. That is

\[ \sum_{i=1}^{k} i^3 = \left( \sum_{i=1}^{k} i \right)^2 \]

Then, for N = k+1.

\[ \sum_{i=1}^{k+1} i^3 = (k + 1)^3 + \sum_{i=1}^{k} i^3 \]

\[ = (k + 1)^3 + \left( \sum_{i=1}^{k} i \right)^2 \]

\[ = (k + 1)^3 + \frac{k^2(k + 1)^2}{4} \]

\[ = \frac{4(k + 1)^3 + k^2(k + 1)^2}{4} \]

\[ = \frac{(k + 1)^2(4k + 4 + k^2)}{4} \]

\[ = \frac{(k + 1)^2(k + 2)^2}{4} \]

\[ = \frac{(k + 1)^2(k + 1 + 1)^2}{4} \]

\[ = \left( \sum_{i=1}^{k+1} i \right)^2 \]

Q.E.D
3.

\(2/N, 37, \sqrt{N}, N, N (\log \log N), N \log N = N \log(N^2), N \log^2 N, N^{1.5}, N^2, N^2 \log N, N^3, 2^{N/2}, 2^N.\)

\(N \log N = N \log(N^2) = 2N \log N \) grow at the same rate
4.

We have,

\[ T(N) = T(N/2) + N^2 \]  for \( N \geq 2 \) and 
\[ T(1) = 0 \]

then,

\[ T(2) = 4, \quad T(4) = 4 + 16 = 20 \]
\[ T(8) = 64, \quad T(16) = 340 \]

We know,

\( f(x) = O(g(x)) \) if and only if there exists a positive real number \( M \) and a real number \( x_0 \) such that

\[ |f(x)| \leq M|g(x)| \text{ for all } x \geq x_0. \]

for \( M = 1 \) and \( x_0 = 2 \) and \( g(x) \) is \( O(N^2) \)

\[ T(2) \leq M \left( 2^{2} \right) = 1 \times 4 = 4, \text{ true} \]
\[ T(4) \leq M \left( 4^{2} \right) = 1 \times 16 = 16, \text{ false, because } T(4) = 20 \]

Again,

\( \text{for } M = 2 \) and \( x_0 = 2 \) and \( g(x) \) is \( O(N^2) \)

\[ T(2) \leq M \left( 2^{2} \right) = 2 \times 4 = 8, \text{ true} \]
\[ T(4) \leq M \left( 4^{2} \right) = 2 \times 16 = 32, \text{ true} \]
\[ T(8) \leq M \left( 8^{2} \right) = 2 \times 64 = 128, \text{ true} \]

Therefore, it is \( O(N^2) \).
Mathematical derivation: Let us change recurrence relation to simple equation. We have,

\[ T(N) = T(N/2) + N^2 \quad \text{for } N \geq 2 \quad \text{and} \]

In above, equation, replace \( T(N/2) \) by its recurrence relation

\[ T(N) = T(N/2) + (N/2)^2 + \ldots + (N/2^k)^2 + (N/2)^2 + N^2 \]

When \( 2^k = N \)

\[ T(N) = T(N/N) + (N/2^{k-1})^2 + \ldots + (N/2^3)^2 + (N/2^2)^2 + (N/2)^2 + N^2 \]

\[ T(N) = T(1) + (N/2^{k-1})^2 + \ldots + (N/2^3)^2 + (N/2^2)^2 + (N/2)^2 + N^2 \]

\[ T(N) = 0 + (N/2^{k-1})^2 + \ldots + (N/2^3)^2 + (N/2^2)^2 + (N/2)^2 + N^2 \]

This is a geometric series start from \( a=1 \) and having ratio \( r=1/2^2 \) with \( k \) terms

Then using, \( \text{Sum} = a(1 - r^k) / (1-r) \) formula, we get

\[ T(N) = N^2 \left[ 1(1-(1/2^2)^k) / (1 - (1/2^2)) \right] \]

\[ T(N) = 4 \, N^2 \left[ 1 - 1/2^k \right] / 3 \]

\[ T(N) = 4 \, N^2 \left[ 1 - 1/N^2 \right] / 3 \]

\[ T(N) = 4 \, (N^2 -1) / 3 \]

Therefore \( T(N) \) is \( O(N^2) \)
Another:

We have,

\[ T(N) = T(N/2) + N^2 \] for \( N \geq 2 \) and

Replace, \( N \) by \( 2^k \)

Then

\[ T(2^k) = T(2^{k-1}) + (2^k)^2 \]
\[ T(2^{k-1}) = T(2^{k-2}) + (2^{k-1})^2 \]
\[ T(2^{k-2}) = T(2^{k-3}) + (2^{k-2})^2 \]

...  
\[ T(2^2) = T(2^1) + (2^2)^2 \]
\[ T(2^1) = T(2^0) + (2^1)^2 \]
\[ T(2^1) = T(1) + (2^1)^2 = 0 + (2^1)^2 \]

Summing all the equations:

\[ T(2^k) = 0 + (2^1)^2 + (2^2)^2 + \ldots + (2^{k-2})^2 + (2^{k-1})^2 + (2^k)^2 \]
\[ T(2^k) = (2^1)^2 + (2^2)^2 + \ldots + (2^{k-2})^2 + (2^{k-1})^2 + (2^k)^2 \]
\[ T(2^k) = (2^1)^1 + (2^2)^2 + \ldots + (2^{k-2})^2 + (2^{k-1})^2 + (2^k)^k \]

Geometric Series with ratio \( 2^2 \), then

\[ T(2^k) = \sum_{i=1}^{i=k} (2^2)^i , \text{https://en.wikipedia.org/wiki/Geometric_series} \]

\[ T(2^k) = 2^2 [(2^2)^k - 1 ] / ( 2^2 - 1) \]
\[ T(N) = 4 (N^2 - 1) / 3 \]

Therefore \( T(N) \) is \( O(N^2) \)